

# The membrane as a perturbation around string-like configurations

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## Abstract

The bosonic membrane in a partial gauge, where one space dimension is eliminated, is formulated as a perturbation theory around an exact free string-like solution. This perturbative regime corresponds to a situation where one of the world-volume space-like dimensions is much greater than the other, so that the membrane has the form of a narrow band or large hoop with string excitations being transverse to the widest dimension. The perturbative equations of motion are studied and solved to first order. Furthermore, it is shown for the open or semi-open cases and to any order in perturbation theory, that one may find canonical transformations that will transform the membrane Hamiltonian into a free string-like Hamiltonian and a boundary Hamiltonian. Thus the membrane dynamics in our perturbation scheme is essentially captured by an interacting boundary theory defined on a two-dimensional world-sheet. A possible implication of this to M-theory is discussed.

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# 1 Introduction

The relativistic membrane is known to be a complicated theory even at the classical level. It is highly non-linear and there are only a very limited number of exact solutions known. At the quantum level still less is known. One of the more promising approaches to the problem is the approximation of the membrane as a matrix model [1–3]. In this approximation one may establish classically that the supermembrane in the light-cone gauge is the large  $N$  limit of the maximally supersymmetric  $SU(N)$  matrix model. At the quantum level such a theory may only be consistent in  $D = 11$ . This can be seen from the fact that, by double dimensional reduction [4], the supermembrane contains the superstring, which is only consistent in ten dimensions. A similar argument tells us that the bosonic membrane is expected to have a critical dimension of 27.

The supermembrane is believed to be related to the conjectured M-theory. Indeed the most fruitful definition hitherto of M-theory is precisely the large  $N$  limit of the above mentioned  $SU(N)$  matrix theory, a relation first conjectured in [5]. This definition, therefore, essentially identifies M-theory and the  $D = 11$  supermembrane. Of course the discretization approach depends crucially on establishing that the continuum limit exists and is well-defined.

It would be useful to be able to analyze the supermembrane or M-theory directly in the continuum. The standard approaches, so far, have been either double-dimensional reduction, yielding a superstring theory, or by taking the field theory limit, in which  $D = 11$  supergravity emerges. The former of these approaches may be used as a definition of M-theory (for early examples see eg. [6,7]). None of these approximations really deal with the full supermembrane degrees of freedom. In this work we will propose an approximation scheme which does deal with the full world-volume dynamics. This scheme will formulate the

membrane as a perturbation theory around a known solution. For simplicity we will only deal with the bosonic membrane, and for reasons that will become clear in the following, only the open or semi-open cases will be of interest to us.

Our perturbation theory starts from fact that the membrane Hamiltonian, in a certain partial gauge, where reparametrizations in one space parameter are fixed and one space dimension is eliminated, may be put into a form

$$H = H_0 + gH_1. \tag{1.1}$$

Here  $H_0$  is the unperturbed Hamiltonian, which is of the same form as the string Hamiltonian, but with the difference that the coordinates  $X^\mu(\xi^a)$ ,  $\mu = 0, 1, \dots, D-2$ , live on the world-volume i.e. depend on three parameters  $\xi^a$ ,  $a = 0, 1, 2$ .  $H_1$  is the perturbation controlled by the parameter  $g$ . As the unperturbed Hamiltonian corresponds to a solvable theory, one may use standard perturbation theory to find the solution to any order.

The perturbative expansion is thus around string-like solutions, where the membrane has the form of a narrow band or large hoop with string excitations being transverse to the largest dimension. Geometrically, this is not the natural string-like setting as this would correspond to the opposite situation, where the string excitations are along the largest dimension. The limit  $g \rightarrow 0$  corresponds to a string of infinite width/circumference. Equivalently, this corresponds to a limit where the membrane tension is much smaller than the string tension associated with the string-like excitations.

Our approach is related to a double-dimensional reduction, since our gauge choice reduces space-time by one space dimension. It may, therefore, be regarded as a perturbation around this dimensional reduction, turning on the dependence on the third world-volume parameter. However, this is not the standard double-dimensional reduction, as the latter corresponds to compactifying one dimension

and taking the radius to zero. Our case is related to the the dual situation where the radius is very large. In [8–10] perturbative calculations around the double-dimensionally reduced membrane in the light-cone gauge was performed, which in spirit is somewhat related to our approach.

Our main focus will, however, not be to solve the perturbative equations of motion in the most straightforward way. Rather, we will deal with the equations by means of the Hamilton-Jacobi method, using canonical transformations. We will establish the rather surprising result that the Hamiltonian of the partially gauge-fixed bosonic membrane is, to any order in perturbation theory, canonically equivalent to the unperturbed Hamiltonian, i.e. to the string-like Hamiltonian, complemented by a boundary Hamiltonian. Thus, in this perturbation scheme the membrane dynamics decomposes into to a free "wide" string and a complicated interacting theory living on the end-lines of this string. These end-lines sweep out two-dimensional world-sheets as they evolve in time.

The paper is organized as follows. In the next section the perturbation theory is established. In section three, the classical solution to the membrane equations of motion is given to first order using straightforward perturbation theory. Construction of the solution by means of canonical transformations is done in section four and, finally, in section five a discussion of our results and possible implications are presented.

## 2 The basic formalism

Let us start from the following Dirac action [11] for the membrane

$$S = -T_{\text{membrane}} \int d^3\xi \left[ -\det \left( \partial_a X_U \partial_b X^U \right) \right]^{1/2}, \quad (2.1)$$

where we use the 'mostly plus' convention on the metric and  $U = 0, \dots, D-1$ ,  $a = 0, 1, 2$ .  $\xi^a$  parametrise the worldvolume with  $\xi^0 = \tau$  being the time component.  $T_{membrane}$  is the membrane tension. In passing to the Hamiltonian one finds the following first class constraints

$$\begin{aligned}\phi_1 &= \mathcal{P}\partial_1 X \approx 0 \\ \phi_2 &= \frac{1}{2} \left\{ \mathcal{P}^2 + T_{membrane}^2 \left[ (\partial_1 X)^2 (\partial_2 X)^2 - (\partial_1 X \partial_2 X)^2 \right] \right\} \approx 0 \\ \phi_3 &= \mathcal{P}\partial_2 X \approx 0,\end{aligned}\tag{2.2}$$

where  $\mathcal{P}_U$  is the canonical momentum conjugate to  $\dot{X}^U$ . Let us fix a partial gauge,

$$\chi \equiv X^{D-1} - \frac{T_{string}}{T_{membrane}} \xi^2 \approx 0,\tag{2.3}$$

where  $T_{string}$  is a constant which can be identified with a string-like tension along the  $\xi^1$  direction. This gauge fixing may be used together with the constraint  $\phi_3 \approx 0$  to eliminate  $X^{D-1}$  and its conjugate momentum. Then one can write the remaining constraints as

$$\begin{aligned}\phi_1 &= \mathcal{P}\partial_1 X \approx 0 \\ \phi_2 &= \frac{1}{2} \left\{ \mathcal{P}^2 + T_{string}^2 (\partial_1 X)^2 \right. \\ &\quad \left. + T_{membrane}^2 \left[ (\partial_1 X)^2 (\partial_2 X)^2 - (\partial_1 X \partial_2 X)^2 + \frac{1}{T_{string}^2} (\mathcal{P}\partial_2 X)^2 \right] \right\} \approx 0,\end{aligned}\tag{2.4}$$

where from now on the scalar products are in  $D-1$  dimensions,  $\mathcal{P}\partial_1 X \equiv \mathcal{P}_\mu \partial_1 X^\mu$ ,  $\mu = 0, \dots, D-2$  etc. Let us fix  $T_{string} = 1$  and take  $T_{membrane} \ll T_{string} = 1$ . Introduce a new parameter  $g \equiv (T_{membrane})^2 \ll 1$ . By writing the membrane tension  $T_{membrane} = T_{string}(L_2)^{-1}$ , where  $L_2$  gives the size in the  $\xi^2$  direction, we see that  $L_2 \gg 1$ . Our constraints now read

$$\phi_1 = \mathcal{P}\partial_1 X \approx 0$$

$$\begin{aligned}\phi_2 &= \frac{1}{2} \left\{ \mathcal{P}^2 + (\partial_1 X)^2 \right. \\ &\quad \left. + g \left[ (\partial_1 X)^2 (\partial_2 X)^2 + (\mathcal{P} \partial_2 X)^2 - (\partial_1 X \partial_2 X)^2 \right] \right\} \approx 0.\end{aligned}\quad (2.5)$$

We then see that  $g$  may be used to define a perturbation theory. In the limit  $g = 0$  the constraints  $\phi_1 \approx 0$  and  $\phi_2 \approx 0$  reduce to the conventional string constraints, with the difference that  $X^\mu$  and  $\mathcal{P}^\mu$  depend on an additional parameter  $\xi^2$ . By using the fundamental Poisson bracket,

$$\{X^\mu(\xi), \mathcal{P}^\nu(\xi')\} = \eta^{\mu\nu} \delta^2(\xi - \xi'), \quad (2.6)$$

one can determine that the constraints satisfy a closed algebra,

$$\begin{aligned}\{\phi_1(\xi), \phi_1(\xi')\} &= (\phi_1(\xi) + \phi_1(\xi')) \partial_1 \delta^2(\xi - \xi') \\ \{\phi_1(\xi), \phi_2(\xi')\} &= (\phi_2(\xi) + \phi_2(\xi')) \partial_1 \delta^2(\xi - \xi') + g \mathcal{P} \partial_2 X \phi_1(\xi') \partial_2 \delta^2(\xi - \xi') \\ \{\phi_2(\xi), \phi_2(\xi')\} &= (\phi_1(\xi) + \phi_1(\xi')) \partial_1 \delta^2(\xi - \xi') \\ &\quad + g \left\{ \left[ (\partial_2 X)^2 \phi_1(\xi) + (\partial_2 X)^2 \phi_1(\xi') \right] \partial_1 \delta^2(\xi - \xi') \right. \\ &\quad - \left[ \partial_1 X \partial_2 X \phi_1(\xi) + \partial_1 X \partial_2 X \phi_1(\xi') \right] \partial_2 \delta^2(\xi - \xi') \\ &\quad \left. + 2 \left[ \mathcal{P} \partial_2 X \phi_2(\xi) + \mathcal{P} \partial_2 X \phi_2(\xi') \right] \partial_2 \delta^2(\xi - \xi') \right\}.\end{aligned}\quad (2.7)$$

We define a Hamiltonian,  $H$ , by the second constraint and separate it into two parts,

$$H_0 = \frac{1}{2} \int d^2 \xi \left[ \mathcal{P}^2 + (\partial_1 X)^2 \right], \quad (2.8)$$

$$H_1 = \frac{1}{2} \int d^2 \xi \left[ (\partial_1 X)^2 (\partial_2 X)^2 + (\mathcal{P} \partial_2 X)^2 - (\partial_1 X \partial_2 X)^2 \right], \quad (2.9)$$

where  $H = H_0 + gH_1$ . Thus  $gH_1$  may be treated as a small perturbation. The gauge fixing, defined in equation (2.3), inserted in  $\phi_3 = 0$  yields

$$\mathcal{P}^{D-1} = -\sqrt{g} \mathcal{P}_\mu \partial_2 X^\mu, \quad (2.10)$$

so that  $\mathcal{P}^{D-1} \rightarrow 0$  when  $g \rightarrow 0$ . Notice also that by (2.3)  $\partial_2 X^{D-1} \rightarrow \infty$  in the same limit.

Let us for completeness show that the perturbation scheme also holds for the BRST extended formalism. We introduce two ghosts  $c^i$  and two anti-ghosts  $b_i$  with the Poisson bracket

$$\left\{c^i(\xi), b_j(\xi')\right\}_+ = \delta_j^i \delta^2(\xi - \xi'). \quad (2.11)$$

One may define a classical BRST charge in the standard way,

$$\begin{aligned} Q_{BRST} = & \int d^2\xi \left\{ \phi_1 c^1 + \phi_2 c^2 + \partial_1 c^1 c^1 b_1 + \partial_1 c^2 c^2 b_1 + \partial_1 c^2 c^1 b_2 + \partial_1 c^1 c^2 b_2 \right. \\ & + g \left[ \mathcal{P} \partial_2 X \partial_2 c^1 c^2 b_1 - \partial_1 X \partial_2 X \partial_2 c^2 c^2 b_1 + (\partial_2 X)^2 \partial_1 c^2 c^2 b_1 \right. \\ & \left. \left. + 2\mathcal{P} \partial_2 X \partial_2 c^2 c^2 b_2 + 2\partial_2 c^1 \partial_2 c^2 c^2 b_1 b_2 \right] \right\}. \end{aligned} \quad (2.12)$$

From this charge one can define a BRST invariant Hamiltonian,

$$\begin{aligned} H = & \int d^2\xi \{Q_{BRST}, b_2\} \\ = & \int d^2\xi \left\{ \phi_2 - \partial_1 c^2 b_1 - \partial_1 c^1 b_2 + g \left[ \partial_1 X \partial_2 X \partial_2 c^2 b_1 - \mathcal{P} \partial_2 X \partial_2 c^1 b_1 \right. \right. \\ & \left. \left. - (\partial_2 X)^2 \partial_1 c^2 b_1 - 2\mathcal{P} \partial_2 X \partial_2 c^2 b_2 + 2\partial_2 c^1 \partial_2 c^2 b_1 b_2 \right] \right\}. \end{aligned} \quad (2.13)$$

We see that the part of the Hamiltonian which contains ghosts breaks into a sum of an unperturbed and perturbed part, just as the non-ghost Hamiltonian did.

### 3 Straightforward perturbation theory

Let us study the equations of motion for the Hamiltonian defined by (2.8) and (2.9),

$$\dot{X}^\mu = \mathcal{P}^\mu + g \partial_2 X^\mu (\mathcal{P} \partial_2 X), \quad (3.1)$$

$$\begin{aligned} \dot{P}^\mu = & \partial_1^2 X^\mu + g \left\{ \partial_1 \left[ \partial_1 X^\mu (\partial_2 X)^2 \right] + \partial_2 \left[ \partial_2 X^\mu (\partial_1 X)^2 \right] + \partial_2 (\mathcal{P}^\mu (\mathcal{P} \partial_2 X)) \right. \\ & \left. - \partial_1 (\partial_2 X^\mu (\partial_1 X \partial_2 X)) - \partial_2 (\partial_1 X^\mu (\partial_1 X \partial_2 X)) \right\}. \end{aligned} \quad (3.2)$$

To solve these equations one can make an expansion of the fields in terms of the perturbation parameter,

$$\begin{aligned} X^\mu &= \sum_{m=0}^{\infty} g^m X_{(m)}^\mu \\ \mathcal{P}^\mu &= \sum_{m=0}^{\infty} g^m \mathcal{P}_{(m)}^\mu. \end{aligned} \quad (3.3)$$

Inserting this into (3.1), (3.2) and separating the equation order by order yields to zeroth order the ordinary string equations of motion,

$$\dot{X}_{(0)}^\mu = \mathcal{P}_{(0)}^\mu \quad (3.4)$$

$$\dot{\mathcal{P}}_{(0)}^\mu = \partial_1^2 X_{(0)}^\mu. \quad (3.5)$$

with the solution,

$$X_{(0)}^\mu = X_R^\mu(\xi^0 - \xi^1, \xi^2) + X_L^\mu(\xi^0 + \xi^1, \xi^2). \quad (3.6)$$

At first order the equations of motion are

$$\dot{X}_{(1)}^\mu = \mathcal{P}_{(1)}^\mu + \partial_2 X_{(0)}^\mu (\mathcal{P}_{(0)} \partial_2 X_{(0)}) \quad (3.7)$$

$$\begin{aligned} \dot{\mathcal{P}}_{(1)}^\mu &= \partial_1^2 X_{(1)}^\mu + \left\{ \partial_1 \left[ \partial_1 X_{(0)}^\mu \left( \partial_2 X_{(0)} \right)^2 \right] + \partial_2 \left[ \partial_2 X_{(0)}^\mu \left( \partial_1 X_{(0)} \right)^2 \right] \right. \\ &+ \partial_2 \left( \mathcal{P}_{(0)}^\mu (\mathcal{P}_{(0)} \partial_2 X_{(0)}) \right) - \partial_1 \left( \partial_2 X_{(0)}^\mu (\partial_1 X_{(0)} \partial_2 X_{(0)}) \right) \\ &\left. - \partial_2 \left( \partial_1 X_{(0)}^\mu (\partial_1 X_{(0)} \partial_2 X_{(0)}) \right) \right\}. \end{aligned} \quad (3.8)$$

Eliminating  $\mathcal{P}^\mu$  from this equation yields

$$\begin{aligned} \square X_{(1)}^\mu &= - \left\{ \partial_0 \left[ \partial_2 X_{(0)}^\mu (\dot{X}_{(0)} \partial_2 X_{(0)}) \right] + \partial_1 \left[ \partial_1 X_{(0)}^\mu \left( \partial_2 X_{(0)} \right)^2 \right] \right. \\ &+ \partial_2 \left[ \partial_2 X_{(0)}^\mu \left( \partial_1 X_{(0)} \right)^2 \right] + \partial_2 \left[ \dot{X}_{(0)}^\mu (\dot{X}_{(0)} \partial_2 X_{(0)}) \right] \\ &\left. - \partial_1 \left[ \partial_2 X_{(0)}^\mu (\partial_1 X_{(0)} \partial_2 X_{(0)}) \right] - \partial_2 \left[ \partial_1 X_{(0)}^\mu (\partial_1 X_{(0)} \partial_2 X_{(0)}) \right] \right\}, \end{aligned} \quad (3.9)$$



where  $\square = -\partial_0^2 + \partial_1^2$ . To solve these equations one can introduce a Greens function,  $\square G_S(\xi, \xi') = \delta^2(\xi - \xi')$ , where the explicit form of  $G_S(\xi, \xi')$  depends on the boundary conditions. One will get the solution by an integration,

$$\begin{aligned}
X^\mu &= \tilde{X}_R^\mu(\xi^0 - \xi^1, \xi^2) + \tilde{X}_L^\mu(\xi^0 + \xi^1, \xi^2) \\
&- \int d^2\xi' G_S(\xi, \xi') \left\{ \partial_0 \left( \partial_2 X_{(0)}^\mu (\dot{X}_{(0)} \partial_2 X) \right) + \partial_1 \left[ \partial_1 X_{(0)}^\mu \left( \partial_2 X_{(0)} \right)^2 \right] \right. \\
&+ \partial_2 \left[ \partial_2 X_{(0)}^\mu \left( \partial_1 X_{(0)} \right)^2 \right] + \partial_2 \left( \dot{X}_{(0)}^\mu (\dot{X}_{(0)} \partial_2 X_{(0)}) \right) \\
&- \left. \partial_1 \left( \partial_2 X_{(0)}^\mu (\partial_1 X_{(0)} \partial_2 X_{(0)}) \right) - \partial_2 \left( \partial_1 X_{(0)}^\mu (\partial_1 X_{(0)} \partial_2 X_{(0)}) \right) \right\} (\xi'),
\end{aligned} \tag{3.10}$$

where  $\tilde{X}_{L/R}^\mu$  is the solution to the homogeneous differential equation.  $\mathcal{P}_{(1)}^\mu$  can be calculated using (3.7). One can proceed in this way to any order which will give us the exact solution to the equations of motion for the membrane. In the next section we will instead of solving the equations of motion directly show that one can use the Hamilton-Jacobi approach by successive canonical transformations. This is extensively studied in the remaining part of this work.

## 4 Solution by canonical transformations

In this section we will deal with the perturbative problem by means of canonical transformations. Our approach is to find a canonical transformation, which will transform away the perturbative corrections, thus solving the equations of motion.

Let us begin by making a change of variables,

$$\begin{aligned}
\alpha^\mu &= \frac{1}{\sqrt{2}} (\mathcal{P}^\mu + \partial_1 X^\mu) \\
\tilde{\alpha}^\mu &= \frac{1}{\sqrt{2}} (\mathcal{P}^\mu - \partial_1 X^\mu),
\end{aligned} \tag{4.1}$$

which diagonalizes the unperturbed Hamiltonian

$$H_0 = \frac{1}{2} \int d^2 \xi \left( \alpha^2 + \tilde{\alpha}^2 \right). \quad (4.2)$$

We have the Poisson bracket relations

$$\begin{aligned} \{\alpha^\mu(\xi), \alpha^\nu(\xi')\} &= \eta^{\mu\nu} \partial_1 \delta^2(\xi - \xi') \\ \{\tilde{\alpha}^\mu(\xi), \tilde{\alpha}^\nu(\xi')\} &= -\eta^{\mu\nu} \partial_1 \delta^2(\xi - \xi') \\ \{\alpha^\mu(\xi), \tilde{\alpha}^\nu(\xi')\} &= 0. \end{aligned} \quad (4.3)$$

These variables are associated with the right- and left-moving modes of the string-like configuration. In order to express  $H_1$  in terms of the new variables, we introduce a Greens function  $K(\xi^1, \xi'^1)$  with these properties

$$\partial_1 K(\xi^1, \xi'^1) = -\partial'_1 K(\xi^1, \xi'^1) = \delta(\xi^1 - \xi'^1). \quad (4.4)$$

Let us denote its operation by  $\partial_1^{-1}$ ,

$$\partial_1^{-1} F(\xi) = \int d\xi'^1 K(\xi^1, \xi'^1) F(\xi^0, \xi'^1, \xi^2). \quad (4.5)$$

It is uniquely defined up to an arbitrary  $\xi^1$ -independent function. From the properties of the Greens function we have  $\partial_1 \partial_1^{-1} F(\xi) = F(\xi)$  and  $\partial_1^{-1} \partial_1 F(\xi) = F(\xi) + f(\xi^0, \xi^2)$ . If  $F(\xi)$  is a periodic function then, in general,  $\partial_1^{-1} F(\xi)$  is not. This implies that a term  $\oint d\xi^1 \partial_1(\dots)$  will not necessarily be zero if the integrand contains terms with  $\partial_1^{-1}$ . This problem is basically the reason why our results do not hold for the closed membrane.

Define the combination  $A = \partial_2 \partial_1^{-1}$  which yields  $\partial_2 X^\mu = \frac{1}{\sqrt{2}}(A\alpha - A\tilde{\alpha})$  and

$$\begin{aligned} \{\alpha^\mu(\xi), A\alpha^\nu(\xi')\} &= \eta^{\mu\nu} \partial_2 \delta^2(\xi - \xi') \\ \{\tilde{\alpha}^\mu(\xi), A\tilde{\alpha}^\nu(\xi')\} &= -\eta^{\mu\nu} \partial_2 \delta^2(\xi - \xi') \\ \{\alpha^\mu(\xi), A\tilde{\alpha}^\nu(\xi')\} &= 0 \\ \{\tilde{\alpha}^\mu(\xi), A\alpha^\nu(\xi')\} &= 0. \end{aligned} \quad (4.6)$$

Inserting the change of variables into  $H_1$  yields

$$\begin{aligned}
H_1 &= \frac{1}{8} \int d^2\xi \left[ \alpha^2 (A\alpha)^2 + \tilde{\alpha}^2 (A\tilde{\alpha})^2 + \tilde{\alpha}^2 (A\alpha)^2 + \alpha^2 (A\tilde{\alpha})^2 \right. \\
&\quad - 2\alpha^2 (A\alpha A\tilde{\alpha}) - 2\tilde{\alpha}^2 (A\alpha A\tilde{\alpha}) - 2(\alpha\tilde{\alpha}) (A\alpha)^2 - 2(\alpha\tilde{\alpha}) (A\tilde{\alpha})^2 \\
&\quad + 4(\alpha\tilde{\alpha}) (A\alpha A\tilde{\alpha}) - 4(\alpha A\alpha) (\tilde{\alpha} A\tilde{\alpha}) - 4(\alpha A\tilde{\alpha}) (\tilde{\alpha} A\alpha) \\
&\quad \left. + 4(\alpha A\alpha) (\tilde{\alpha} A\alpha) + 4(\alpha A\tilde{\alpha}) (\tilde{\alpha} A\tilde{\alpha}) \right]. \tag{4.7}
\end{aligned}$$

Let us investigate if one can make a canonical transformation  $(\alpha^\mu, \tilde{\alpha}^\mu) \rightarrow (\alpha'^\mu, \tilde{\alpha}'^\mu)$  such that  $H_0 + gH_1 \rightarrow H_0$  to first order in perturbation theory, i.e. we want a canonical transformation with the following property,

$$\begin{aligned}
H_0(\alpha'^\mu, \tilde{\alpha}'^\mu) &= H_0(\alpha^\mu, \tilde{\alpha}^\mu) + g \{H_0(\alpha^\mu, \tilde{\alpha}^\mu), G_1\} \\
&= H_0(\alpha^\mu, \tilde{\alpha}^\mu) + gH_1(\alpha^\mu, \tilde{\alpha}^\mu). \tag{4.8}
\end{aligned}$$

One can, by inspection, directly find an expression for  $G_1$  for the terms that do not mix  $\alpha^\mu$  and  $\tilde{\alpha}^\mu$ . Assume that we have the following term in the perturbed Hamiltonian,

$$H_1^{part} = \int d^2\xi C_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A\alpha^{\mu_{m+1}} \dots A\alpha^{\mu_n}, \tag{4.9}$$

where  $C_{\mu_1 \dots \mu_n}$  is a constant tensor. By inspection one finds that

$$\tilde{G}_1^{part} = \int d^2\xi C_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A\alpha^{\mu_{m+1}} \dots A\alpha^{\mu_n} (\xi) \xi^1, \tag{4.10}$$

solves equation (4.8) up to boundary terms. This solution can be written as (with a suitable choice of integration constant)

$$\begin{aligned}
\tilde{G}_1^{part} &= \int d^2\xi C_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A\alpha^{\mu_{m+1}} \dots A\alpha^{\mu_n} (\xi) \\
&\quad \cdot \int d\xi'^1 K_A(\xi^1, \xi'^1), \tag{4.11}
\end{aligned}$$

where  $K_A(\xi^1, \xi'^1)$  is defined by

$$K_A(\xi^1, \xi'^1) = \frac{1}{2} \left[ K(\xi^1, \xi'^1) - K(\xi'^1, \xi^1) \right]. \tag{4.12}$$

This follows from the explicit form of the Greens function. There also exists another solution

$$G_1^{part} = - \int d^2\xi \int d\xi'^1 K_A(\xi^1, \xi'^1) C_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_n}(\xi^0, \xi'^1, \xi^2) A\alpha^{\mu_{m+1}} \dots A\alpha^{\mu_n}(\xi^0, \xi'^1, \xi^2). \quad (4.13)$$

In the same manner a term of the form  $H_1^{part}(\xi) = \int d^2\xi C_{\mu_1 \dots \mu_n} \tilde{\alpha}^{\mu_1} \dots \tilde{\alpha}^{\mu_m} \tilde{A}\alpha^{\mu_{m+1}} \dots \tilde{A}\alpha^{\mu_n}$  is solved by

$$G_1^{part} = - \int d^2\xi C_{\mu_1 \dots \mu_n} \tilde{\alpha}^{\mu_1} \dots \tilde{\alpha}^{\mu_m} A\tilde{\alpha}^{\mu_{m+1}} \dots A\tilde{\alpha}^{\mu_n}(\xi) \cdot \int d\xi'^1 K_A(\xi^1, \xi'^1) \quad (4.14)$$

or

$$\tilde{G}_1^{part} = \int d^2\xi \int d\xi'^1 K_A(\xi^1, \xi'^1) C_{\mu_1 \dots \mu_n} \tilde{\alpha}^{\mu_1} \dots \tilde{\alpha}^{\mu_m}(\xi^0, \xi'^1, \xi^2) \tilde{A}\alpha^{\mu_{m+1}} \dots \tilde{A}\alpha^{\mu_n}(\xi^0, \xi'^1, \xi^2), \quad (4.15)$$

where (4.14) can be written as (up to a suitable integration constant)

$$G_1^{part} = - \int d^2\xi C_{\mu_1 \dots \mu_n} \tilde{\alpha}^{\mu_1} \dots \tilde{\alpha}^{\mu_m} A\tilde{\alpha}^{\mu_{m+1}} \dots A\tilde{\alpha}^{\mu_n}(\xi) \xi^1.$$

For the parts that mix different kinds of modes the situation is more complicated.

Consider a general term of the type

$$H_1^{part} = \int d^2\xi C_{\mu_1 \dots \mu_q} \alpha^{\mu_1} \dots \alpha^{\mu_m} A\alpha^{\mu_{m+1}} \dots A\alpha^{\mu_n} \tilde{\alpha}^{\mu_{n+1}} \dots \tilde{\alpha}^{\mu_p} A\tilde{\alpha}^{\mu_{p+1}} \dots A\tilde{\alpha}^{\mu_q}. \quad (4.16)$$

Let us make the following ansatz for the canonical generator

$$G_1^{part} = -\frac{1}{2} \int d^2\xi C_{\mu_1 \dots \mu_q} \int d\xi'^1 K_A(\xi^1, \xi'^1) \alpha^{\mu_1} \dots \alpha^{\mu_m}(\xi^0, \xi'^1, \xi^2) A\alpha^{\mu_{m+1}} \dots A\alpha^{\mu_n}(\xi^0, \xi'^1, \xi^2) \tilde{\alpha}^{\mu_{n+1}} \dots \tilde{\alpha}^{\mu_p} A\tilde{\alpha}^{\mu_{p+1}} \dots A\tilde{\alpha}^{\mu_q}(\xi), \quad (4.17)$$

where  $m \leq n \leq p \leq q$ ,  $0 < n$  and  $n < q$ . Its Poisson bracket with  $H_0$  is

$$\begin{aligned}
\{H_0, G_1^{part}\} = & -\frac{1}{2} \int d^2\xi \int d^2\xi' C_{\mu_1 \dots \mu_q} \left\{ \int d\xi''^1 K_A(\xi'^1, \xi''^1) \right. \\
& \left[ \sum_{i=1}^m \alpha^{\mu_i}(\xi) \partial_1 \delta(\xi^1 - \xi''^1) \delta(\xi^2 - \xi'^2) \right. \\
& \alpha^{\mu_1} \dots \widehat{\alpha^{\mu_i}} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n}(\xi^0, \xi''^1, \xi'^2) \\
& + \sum_{i=m+1}^n \alpha^{\mu_i}(\xi) \delta(\xi^1 - \xi''^1) \partial_2 \delta(\xi^2 - \xi'^2) \\
& \alpha^{\mu_1} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots \widehat{A \alpha^{\mu_i}} \dots A \alpha^{\mu_n}(\xi^0, \xi''^1, \xi'^2) \Big] \\
& \tilde{\alpha}^{\mu_{n+1}} \dots \tilde{\alpha}^{\mu_p} A \tilde{\alpha}^{\mu_{p+1}} \dots A \tilde{\alpha}^{\mu_q}(\xi^0, \xi'^1, \xi'^2) \\
& - \int d\xi''^1 K_A(\xi'^1, \xi''^1) \alpha^{\mu_1} \dots \alpha^{\mu_m}(\xi^0, \xi''^1, \xi'^2) \\
& A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n}(\xi^0, \xi''^1, \xi'^2) \\
& \left[ \sum_{i=n+1}^p \tilde{\alpha}^{\mu_i}(\xi) \partial_1 \delta^2(\xi - \xi') \tilde{\alpha}^{\mu_{n+1}} \dots \widehat{\tilde{\alpha}^{\mu_i}} \dots \tilde{\alpha}^{\mu_p}(\xi^0, \xi'^1, \xi'^2) \right. \\
& A \tilde{\alpha}^{\mu_{p+1}} \dots A \tilde{\alpha}^{\mu_q}(\xi^0, \xi'^1, \xi'^2) \\
& + \sum_{i=p+1}^q \tilde{\alpha}^{\mu_i}(\xi) \partial_2 \delta^2(\xi - \xi') \tilde{\alpha}^{\mu_{n+1}} \dots \tilde{\alpha}^{\mu_p}(\xi^0, \xi'^1, \xi'^2) \\
& \left. A \tilde{\alpha}^{\mu_{p+1}} \dots \widehat{A \tilde{\alpha}^{\mu_i}} \dots A \tilde{\alpha}^{\mu_q}(\xi^0, \xi'^1, \xi'^2) \right] \Big\},
\end{aligned}$$

where the hat denotes an omitted term. This can be simplified using the relations in eq.(4.4) and (4.12) to read

$$\begin{aligned}
\{H_0, G_1\} = & H_1^{part} \\
& + \frac{1}{2} C_{\mu_1 \dots \mu_q} \left\{ \int d\xi^2 [(m-1) \alpha^{\mu_1} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n}(\xi) \right. \\
& \int d\xi'^1 K_A(\xi^1, \xi'^1) \tilde{\alpha}^{\mu_{n+1}} \dots \tilde{\alpha}^{\mu_p}(\xi^0, \xi'^1, \xi^2) \\
& A \tilde{\alpha}^{\mu_{p+1}} \dots A \tilde{\alpha}^{\mu_q}(\xi^0, \xi'^1, \xi^2) \\
& + (p-n-1) \tilde{\alpha}^{\mu_{n+1}} \dots \tilde{\alpha}^{\mu_p} A \tilde{\alpha}^{\mu_{p+1}} \dots A \tilde{\alpha}^{\mu_q}(\xi) \\
& \left. \int d\xi'^1 K_A(\xi^1, \xi'^1) \alpha^{\mu_1} \dots \alpha^{\mu_m}(\xi^0, \xi'^1, \xi^2) \right\}
\end{aligned}$$

$$A\alpha^{\mu_{m+1}} \cdot \dots \cdot A\alpha^{\mu_n}(\xi^0, \xi'^1, \xi^2) \Big] \Big\}_{\xi^1=0}^{L_1} + \{\dots\}_{\xi^2=0}^{L_2}. \quad (4.18)$$

Our ansatz yields, therefore, the correct term in the Hamiltonian. From this derivation one can also see that boundary terms arise. The solution  $G_1^{part}$  is not the only one. For a Hamiltonian, as in eq.(4.16), there are at least two linearly independent solutions,

$$\begin{aligned} G_1^{part} &= -\frac{1}{2} \int d^2\xi C_{\mu_1 \dots \mu_q} \int d\xi'^1 K_A(\xi^1, \xi'^1) \alpha^{\mu_1} \cdot \dots \cdot \alpha^{\mu_m}(\xi^0, \xi'^1, \xi^2) \\ &\quad A\alpha^{\mu_{m+1}} \cdot \dots \cdot A\alpha^{\mu_n}(\xi^0, \xi'^1, \xi^2) \tilde{\alpha}^{\mu_{n+1}} \cdot \dots \cdot \tilde{\alpha}^{\mu_p}(\xi) \\ &\quad A\tilde{\alpha}^{\mu_{p+1}} \cdot \dots \cdot A\tilde{\alpha}^{\mu_q}(\xi) \end{aligned} \quad (4.19)$$

$$\begin{aligned} \tilde{G}_1^{part} &= \frac{1}{2} \int d^2\xi C_{\mu_1 \dots \mu_q} \alpha^{\mu_1} \cdot \dots \cdot \alpha^{\mu_m} A\alpha^{\mu_{m+1}} \cdot \dots \cdot A\alpha^{\mu_n}(\xi) \\ &\quad \int d\xi'^1 K_A(\xi^1, \xi'^1) \tilde{\alpha}^{\mu_{n+1}} \cdot \dots \cdot \tilde{\alpha}^{\mu_p}(\xi^0, \xi'^1, \xi^2) \\ &\quad A\tilde{\alpha}^{\mu_{p+1}} \cdot \dots \cdot A\tilde{\alpha}^{\mu_q}(\xi^0, \xi'^1, \xi^2) \end{aligned} \quad (4.20)$$

From here on we use the most symmetric linear combination of these two solutions

$$G_1^{sym} = \frac{1}{2} (G_1 + \tilde{G}_1). \quad (4.21)$$

This will, in general, lead to simpler boundary terms. Also, we will consider an open or semi-open membrane. These cases may be solved quite generally in this approach.

Using eqs.(4.11), (4.13)-(4.15) and (4.19)-(4.21) one finds the canonical generator to first order

$$\begin{aligned} G_1^{sym} &= -\frac{1}{32} \int d^2\xi \left[ 2\partial_1^{-1} \left( \alpha^2 (A\alpha)^2 \right) - 2\alpha^2 (A\alpha)^2 \partial_1^{-1}(1) \right. \\ &\quad + 2\tilde{\alpha}^2 (A\tilde{\alpha})^2 \partial_1^{-1}(1) - 2\partial_1^{-1} \left( \tilde{\alpha}^2 (A\tilde{\alpha})^2 \right) \\ &\quad + \tilde{\alpha}^2 \partial_1^{-1} (A\alpha)^2 - \partial_1^{-1} \tilde{\alpha}^2 (A\alpha)^2 + \partial_1^{-1} \alpha^2 (A\tilde{\alpha})^2 - \alpha^2 \partial_1^{-1} (A\tilde{\alpha})^2 \\ &\quad \left. - 2\partial_1^{-1} \left( \alpha^2 A\alpha_\mu \right) A\tilde{\alpha}^\mu + 2\alpha^2 A\alpha_\mu \partial_1^{-1} A\tilde{\alpha}^\mu - 2\tilde{\alpha}^2 A\tilde{\alpha}^\mu \partial_1^{-1} A\alpha_\mu \right] \end{aligned}$$

$$\begin{aligned}
& + 2\partial_1^{-1} \left( \tilde{\alpha}^2 A \tilde{\alpha}^\mu \right) A \alpha_\mu - 2\tilde{\alpha}^\mu \partial_1^{-1} \left( \alpha_\mu (A \alpha)^2 \right) + 2\partial_1^{-1} \tilde{\alpha}^\mu \alpha_\mu (A \alpha)^2 \\
& - 2\partial_1^{-1} \alpha_\mu \tilde{\alpha}^\mu (A \tilde{\alpha})^2 + 2\alpha_\mu \partial_1^{-1} (\tilde{\alpha}^\mu A \tilde{\alpha})^2 + 4\partial_1^{-1} (\alpha_\mu A \alpha_\nu) \tilde{\alpha}^\mu A \tilde{\alpha}^\nu \\
& - 4\alpha_\mu A \alpha_\nu \partial_1^{-1} (\tilde{\alpha}^\mu A \tilde{\alpha}^\nu) - 4\partial_1^{-1} (\alpha_\mu A \alpha^\mu) \tilde{\alpha}_\nu A \tilde{\alpha}^\nu \\
& + 4\alpha_\mu A \alpha^\mu \partial_1^{-1} (\tilde{\alpha}_\nu A \tilde{\alpha}^\nu) - 4\partial_1^{-1} (\alpha_\mu A \alpha^\nu) A \tilde{\alpha}^\mu \tilde{\alpha}_\nu \\
& + 4\alpha_\mu A \alpha^\nu \partial_1^{-1} (A \tilde{\alpha}^\mu \tilde{\alpha}_\nu) + 4\partial_1^{-1} (\alpha_\mu A \alpha^\mu A \alpha^\nu) \tilde{\alpha}_\nu \\
& - 4\alpha_\mu A \alpha^\mu A \alpha^\nu \partial_1^{-1} \tilde{\alpha}_\nu + 4\partial_1^{-1} \alpha_\mu A \tilde{\alpha}^\mu \tilde{\alpha}_\nu A \tilde{\alpha}^\nu \\
& - 4\alpha_\mu \partial_1^{-1} (A \tilde{\alpha}^\mu \tilde{\alpha}_\nu A \tilde{\alpha}^\nu) \Big], \tag{4.22}
\end{aligned}$$

where  $\partial_1^{-1}$  is defined in eq.(4.4), with the arbitrary function set to zero. Thus, what we have shown is that in place of eq.(4.8) we find

$$H_0 + g \{H_0, G_1\} = H_0 + gH_1 + gH_{B1}. \tag{4.23}$$

In order to see the structure of the boundary Hamiltonian  $H_{B1}$ , let us collect the boundary terms that arise from the canonical transformation. A straightforward calculation gives the following explicit expression of the boundary term

$$\begin{aligned}
H_{B1} = & \left\{ \frac{1}{16} \int d\xi^2 \left[ 2\alpha^2 (A \alpha)^2 \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) \right. \right. \\
& + 2\tilde{\alpha}^2 (A \tilde{\alpha})^2 \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) \\
& + \alpha^2 \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) (A \tilde{\alpha})^2 (\xi^0, \xi'^1, \xi^2) \\
& - (A \tilde{\alpha})^2 \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) \alpha^2 (\xi^0, \xi'^1, \xi^2) \\
& + \tilde{\alpha}^2 \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) (A \alpha)^2 (\xi^0, \xi'^1, \xi^2) \\
& - (A \alpha)^2 \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) \tilde{\alpha}^2 (\xi^0, \xi'^1, \xi^2) \\
& - 2\alpha^2 A \alpha_\mu \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) A \tilde{\alpha}^\mu (\xi^0, \xi'^1, \xi^2) \\
& + 2A \tilde{\alpha}_\mu \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) \alpha^2 A \alpha^\mu (\xi^0, \xi'^1, \xi^2) \\
& \left. \left. - 2\tilde{\alpha}^2 A \tilde{\alpha}_\mu \int d\xi'^1 K_A \left( \xi^1, \xi'^1 \right) A \alpha^\mu (\xi^0, \xi'^1, \xi^2) \right] \right\}
\end{aligned}$$

$$+ 2A\alpha_\mu \int d\xi'^1 K_A(\xi^1, \xi'^1) \tilde{\alpha}^2 A\tilde{\alpha}^\mu(\xi^0, \xi'^1, \xi^2) \Big] \Big\}_{\xi^1=0}^{L_1}. \quad (4.24)$$

Notice that the boundary Hamiltonian is non-local, since it involves several integrated Greens functions. Therefore, it is not a boundary in the strict sense that it depends on local functions of the fields at the boundary. Still it is a boundary Hamiltonian in the sense that the corresponding Hamiltonian density is everywhere equal to its boundary value.

Let us make some comments. We have shown above  $H(\alpha, \tilde{\alpha}) + H_{B1}(\alpha, \tilde{\alpha}) = H_0(\alpha', \tilde{\alpha}') + O(g^2)$ . This implies, however,

$$H(\alpha, \tilde{\alpha}) = H_0(\alpha', \tilde{\alpha}') - H_{B1}(\alpha', \tilde{\alpha}') + O(g^2). \quad (4.25)$$

This means that the original partially gauge-fixed membrane Hamiltonian is decomposed into a string-like Hamiltonian and a complicated boundary Hamiltonian. This conclusion is also true to any finite order in perturbation theory, as we will show.

Our results include the semi-open case. By choice, we can always take the boundary to have fixed  $\xi^1$  i.e. the string-like membrane along the  $\xi^1$ -direction is still open and the two-dimensional surface a boundary is closed in the space-direction. Alternatively, we may mix the two cases having one of each type at the two boundaries. For the fully closed membrane the situation is different because, in this case, there do not exist any Greens functions that satisfies eq(4.4).

Let us now show that one may extend the result to all orders in perturbation theory. To second order the canonical transformation generated by  $G_1$  is

$$f' = f + g \{f, G_1\} + \frac{g^2}{2} \{\{f, G_1\}, G_1\}, \quad (4.26)$$

If we set  $f = H_0$  one can see that we generate a new bulk- and boundary<sup>3</sup>-

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<sup>3</sup>From  $G_1$  we generate new boundary terms of order  $g^2$



Hamiltonian,

$$\begin{aligned} H_0(\alpha'^\mu, \tilde{\alpha}'^\mu) &= H_0(\alpha^\mu, \tilde{\alpha}^\mu) + g[H_1(\alpha^\mu, \tilde{\alpha}^\mu) + H_{B1}(\alpha^\mu, \tilde{\alpha}^\mu)] \\ &+ \frac{1}{2}g^2[H_2(\alpha^\mu, \tilde{\alpha}^\mu) + H_{B2}(\alpha^\mu, \tilde{\alpha}^\mu)] + O(g^3). \end{aligned} \quad (4.27)$$

This can be rewritten as

$$\begin{aligned} H_0(\alpha'^\mu, \tilde{\alpha}'^\mu) + g^2 H_2(\alpha'^\mu, \tilde{\alpha}'^\mu) &= H_0(\alpha^\mu, \tilde{\alpha}^\mu) + g H_1(\alpha^\mu, \tilde{\alpha}^\mu) + \\ &\text{boundary terms} + O(g^3). \end{aligned} \quad (4.28)$$

We now make a new canonical transformation generated by  $G_2$  such that it compensates for the term  $H_2$ . This procedure can be continued. To  $N$ 'th order, where  $N > 1$ , one deduces this equation for the generator  $G_N$

$$\{H_0, G_N\} = - \sum_{m,n \in \mathbb{N}, m \neq 1, nm=N} (-1)^m \frac{1}{m!} \text{ad}_{G_n}^m(H_0), \quad (4.29)$$

where  $\text{ad}_{G_n}^m f \equiv \{G_n, \dots, \{G_n, \{G_n, f\} \dots\}$  ( $m$  brackets). The most general bulk term of order  $N$  in the Hamiltonian is of the form

$$H_N^{\text{part}} = \int d\xi^2 d\sigma C(\sigma_1, \dots, \sigma_n) \alpha(\sigma_1) \cdot \dots \cdot \alpha(\sigma_m) \tilde{\alpha}(\sigma_{m+1}) \cdot \dots \cdot \tilde{\alpha}(\sigma_n), \quad (4.30)$$

where we have suppressed the index structure (cf. eq.(4.9)) and for  $\alpha$  and  $\tilde{\alpha}$  we have written

$$\alpha(\sigma_i) = \alpha(\xi^0, \sigma_i, \xi^2). \quad (4.31)$$

Let us make the ansatz

$$\begin{aligned} G_N^{\text{part}} &= \int d\xi^2 d\sigma d\rho F(\sigma_1, \dots, \sigma_n, \rho_1, \dots, \rho_n) C(\rho_1, \dots, \rho_n) \\ &\alpha(\sigma_1) \cdot \dots \cdot \alpha(\sigma_m) \tilde{\alpha}(\sigma_{m+1}) \cdot \dots \cdot \tilde{\alpha}(\sigma_n). \end{aligned} \quad (4.32)$$

The Poisson bracket between this ansatz and  $H_0$  is,

$$\begin{aligned} \{H_0, G_N^{part}\} &= \int d\xi^2 d\sigma d\rho \left[ \sum_{i=1}^m \frac{\partial}{\partial \sigma_i} - \sum_{i=m+1}^n \frac{\partial}{\partial \sigma_i} \right] F(\sigma_1, \dots, \sigma_n, \rho_1, \dots, \rho_n) \\ &\quad C(\rho_1, \dots, \rho_n) \alpha(\sigma_1) \cdot \dots \cdot \alpha(\sigma_m) \tilde{\alpha}(\sigma_{m+1}) \cdot \dots \cdot \tilde{\alpha}(\sigma_n). \end{aligned} \quad (4.33)$$

For this to be equal to (4.30) one can see that the function  $F$  has to satisfy

$$\left[ \sum_{i=1}^m \frac{\partial}{\partial \sigma_i} - \sum_{i=m+1}^n \frac{\partial}{\partial \sigma_i} \right] F(\sigma_1, \dots, \sigma_n, \rho_1, \dots, \rho_n) = \prod_{i=1}^n \delta(\sigma_i - \rho_i). \quad (4.34)$$

The simplest way to solve this equation is to make a coordinate transformation,

$$\begin{aligned} \eta_i &= \sum_{j=1}^n B_{ij} \sigma_j \\ \mu_i &= \sum_{j=1}^n B_{ij} \rho_j, \end{aligned} \quad (4.35)$$

such that the matrix  $B_{ij}$  is invertible,  $\det(B_{ij}) = 1$  and satisfies

$$(B^{-1})_{i1} = \begin{cases} 1 & i = 1, \dots, m \\ -1 & i = m+1, \dots, n. \end{cases} \quad (4.36)$$

Inserting this into eq.(4.34) yields

$$\frac{\partial}{\partial \eta_1} \tilde{F}(\eta_1, \dots, \eta_n, \mu_1, \dots, \mu_n) = \prod_{i=1}^n \delta(\eta_i - \mu_i), \quad (4.37)$$

where  $\tilde{F}$  is related to  $F$  by the variable transformation in eq.(4.35). One can now use the Greens function to get the solution

$$\begin{aligned} \tilde{F}(\eta_1, \dots, \eta_n, \mu_1, \dots, \mu_n) &= \int d\eta'_1 K_A(\eta_1, \eta'_1) \delta(\eta'_1 - \mu_1) \\ &\quad \cdot \prod_{i=2}^n \delta(\eta_i - \mu_i). \end{aligned} \quad (4.38)$$

Thus, all different kinds of terms that arise in the perturbative expansion can be solved. An expression for any quantity  $f$  to order  $N$  can be written as

$$f^{(N)} = f + \sum_{n=1}^N \sum_{m=1}^{int(N/n)} (-1)^m \frac{g^{mn}}{m!} \text{ad}_{G_n}^m f, \quad (4.39)$$

where  $int(N/n)$  is the integer part of  $N/n$ . For the Hamiltonian one can deduce that

$$H = H_0(\alpha^{(N)}, \tilde{\alpha}^{(N)}) + O(g^{N+1}) + \text{boundary terms}. \quad (4.40)$$

Certain terms in the perturbative expansion are particularly simple to transform away. These are the terms that involve only one type of modes. For instance, if one looks at the terms involving  $\alpha^\mu$  only, one can use the simple solution in eq.(4.10). This generates terms of this type to second order,

$$H_2^{part} = \int d^2\xi D_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n} (\xi) \xi^1, \quad (4.41)$$

where  $D$  may be an operator acting on the fields. To compensate for this term one can use a canonical transformation generated by

$$\tilde{G}_2^{part} = \int d^2\xi D_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n} (\xi) \frac{(\xi^1)^2}{2}. \quad (4.42)$$

If one proceeds to  $N$ 'th order one finds

$$H_N^{part} = \int d^2\xi D_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n} (\xi) \xi^{N-1}, \quad (4.43)$$

which is solved by

$$\tilde{G}_N^{part} = \int d^2\xi D_{\mu_1 \dots \mu_n} \alpha^{\mu_1} \dots \alpha^{\mu_m} A \alpha^{\mu_{m+1}} \dots A \alpha^{\mu_n} (\xi) \frac{(\xi^1)^N}{N}. \quad (4.44)$$

It is interesting to note that this particularly simple solution may be applied to a string case, which implies that any interaction term in the Hamiltonian may be eliminated classically in a perturbative manner.

## 5 Discussion

Our treatment of the bosonic membrane, formulating it as a perturbation theory around an open string-like solution, has shown to any order in perturbation theory, that the membrane equations of motions may be solved in the bulk of the world-volume by performing canonical transformations transforming the membrane Hamiltonian to the free string-like Hamiltonian. At the two end-lines of the "wide" string there remains complicated interacting theories living on the two two-dimensional world-sheets that are traced out by the end-lines. These world-sheets are either open, closed, or mixed in the space-direction, where the first possibility requires a fully open membrane. Of course, the bulk and boundary theories are not independent. Rather, the dynamics at the two boundaries are mediated by the free string oscillations of the bulk.

It should be pointed out that our analysis here does not imply that the membrane theory is equivalent to a string-like theory together with a boundary theory, as we have only shown that, in a particular gauge, the Hamiltonians are related in this way. In order to complete the picture, we also need to show that the physical subspaces implied by the remaining constraints coincide. One may, in fact, easily realize that the constraints will not be canonically equivalent, not even up to boundary terms. If this would have been the case, then the constraint algebra of the remaining constraints would have to satisfy the Virasoro algebra, which they clearly do not when higher order terms are taken into account, as can be seen from eq.(2.7). This does not necessarily mean that the physical subspaces are inequivalent in the bulk. In order to use canonical transformations to determine if the physical subspaces are equivalent, up to boundary quantities, one would need to extend the treatment using the membrane BRST charge. If one can show that, by extending the canonical transformations to the ghost sector,

the BRST charge of the membrane transforms into the string one and a boundary charge, then we are assured that our results here are also true for the physical solutions of the equations of motion. Notice that such canonical transformations mix the space-time coordinates and ghosts in a highly non-trivial way, as can be seen from the explicit form of the BRST charge (2.12). We hope to be able to report further on this issue in a forthcoming publication.

It should be remarked that from a string point of view the perturbation theory formulated here is highly non-trivial. The canonical transformations impose corrections to the string-like modes  $\alpha_n^\mu$  and  $\tilde{\alpha}_n^\mu$ , that have infinite net mode number, even at first order in perturbation theory. The terms that are responsible for this in the generator are the ones that mix  $\alpha_n^\mu$  and  $\tilde{\alpha}_n^\mu$ . Consequently, the perturbative expansion is non-perturbative from a string point of view.

Our discussion here has been purely classical. The fact that we have formulated our perturbation theory around a free "wide" string means, however, that at least to zeroth order in perturbation theory we have a quantum mechanically consistent starting point<sup>4</sup>, including a vacuum state and other physical states. In particular, our starting point requires the number of space-time dimensions to be 27. Of course, the main challenge is to see whether this is consistent to higher orders in perturbation theory. This requires us to extend the analysis, including the ghosts and finding first a canonical transformation which transforms the BRST charge (2.12), modified by boundary terms, to the string BRST charge, as was discussed above. Then, if this transformation may be extended to a unitary one at the quantum level, consistency at  $D = 27$  is established. It might seem that the hope of showing that the canonical transformation extends into a unitary one is very optimistic. However, the main problem is one of order-

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<sup>4</sup>Disregarding the usual problems of the bosonic string

ing operators and this may be solved by requiring that the ordering defined by consistency of the string-like solution, will define the ordering of the operators prior to transformation.

We have here treated the bosonic membrane. The most interesting case is, however, the supermembrane. It remains to see whether our treatment extends to this case as well. Our belief is that this indeed is the case. There seems to be no principle difference between the two when it comes to formulating the perturbation theory. However, it is well-known that the two cases differ in many respects. One of the more important aspects is that the bosonic membrane has a discrete spectrum, whereas the supermembrane has a continuous one [12]. This has important consequences in the interpretation of the latter theory (see eg. the discussion in [13]). Whether a similar treatment of the supermembrane will highlight these differences remains to be seen. It may turn out that the complexity of the boundary Hamiltonian prevents any further understanding in this respect.

Let us end this discussion with an even more speculative remark. As we have pointed out several times the non-trivial part of the membrane theory, in our scheme, are the boundary theories living on the two-dimensional world-sheets at the end-lines. It would be tempting to say that each of these latter theories correspond to some sort of interacting string theory. For this to be true, it is necessary that the membrane theory, supplemented by boundary theories, are reparametrization invariant also at the boundaries. Taking such a fully invariant theory our perturbative scheme may imply that this theory is canonically equivalent to an interacting open or closed string theory at each boundary, which communicate with each other through the free "wide" string in the bulk. We could go one step further and say that M-theory, in a partial gauge, may perhaps be *defined* in this

way. With such a point of view, M-theory dynamics would essentially reduce to that of two coupled interacting string theories. Such a definition would make it possible to analyze M-theory in great detail.

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